Locality, detection efficiencies, and probability polytopes

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We present a detailed investigation of minimum detection efficiencies, below which locality cannot be violated by any quantum system of any dimension in bipartite Bell experiments. Lower bounds on these minimum detection efficiencies are determined with the help of linear programming techniques. Our approach is based on the observation that any possible bipartite quantum correlation originating from a quantum state in an arbitrary dimensional Hilbert space is sandwiched between two probability polytopes, namely the local (Bell) polytope and a corresponding nonlocal no-signaling polytope. Numerical results are presented demonstrating the dependence of these lower bounds on the numbers of inputs and outputs of the bipartite physical system.

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I. INTRODUCTION

Despite numerous recent experimental tests for violations of locality by quantum theory, such as the experiment by Weihs et al. [1], we still do not know for certain whether or not the laws of physics are entirely local [2]. This is because so far no single experiment has closed both the locality and the detection loophole simultaneously.

In general, classical correlations between two spacelike separated experimenters, Alice (A) and Bob (B), obey locality constraints, which can be expressed in terms of (generalized) Bell inequalities. According to Bell's theorem [3], these inequalities can be violated if the relevant correlations are produced by ideal measurements of an entangled quantum system, whose quantum state is required to originate in the common backward light cone of A and B. This is weak nonlocality. Strong nonlocality would be the non-existent correlations produced by signaling faster than the speed of light.

In a typical two-photon Bell experiment the polarization state of a pair of entangled photons is measured independently by A and B. Preferred states for such experiments are pure two-photon states of maximum entanglement, the so-called Bell states.

One of the last remaining major problems on the way to a loophole-free test of locality is the detection loophole, which comes from photon detection efficiencies being too small [4, 5]. In this context Eberhard [6] recognized that weakly entangled pure two-photon states yield a maximized tolerance to detection inefficiency. Using numerical optimization he was thus able to reduce the critical detection efficiency in the two-photon Bell experiment towards a theoretical limit of 2/3 under the assumption of identical detection efficiencies for A and B. Later authors [4, 5] have applied his method to other experiments of the Bell type and achieved theoretical values of the critical detection efficiency as low as 0.43 for the extreme asymmetric case in which only either A's or B's detection is perfect.

In view of these results on minimum detection efficiencies two major questions arise. Firstly, it is not clear whether these results also apply to Bell experiments in which the dichotomic variables measured by A and B result from quantum observables and quantum states in arbitrarily high dimensional Hilbert spaces. Secondly, it is unclear what influence symmetric and asymmetric detection efficiencies have on cases in which more than two physical quantities are measured on A's and B's sides or on cases in which the observables have more than two possible outcomes. It is the main purpose of this paper to address these open questions.

For this purpose an efficient way is developed for describing local bipartite correlations with the help of probability polytopes and linear programming. It is known that the relevant local polytopes can be described efficiently in terms of their vertexes, which can be obtained for any experimental setup of any number of inputs and outputs, as described here in Sec. II. Thus, any test of locality reduces to an inclusion test determining whether a given set of probabilities is located outside or inside the relevant local polytope. In addition, with the help of a second class of probability polytopes which describe nonlocal no-signaling correlations [7] it is possible to obtain lower bounds on minimum detection efficiencies for bipartite Bell experiments. These latter probability polytopes include all correlations of bipartite quantum systems of any dimension and thus yield dimension-independent lower bounds on detection efficiencies. First results of such lower bounds are presented for inefficiencies of arbitrary symmetry and for bipartite locality tests with dichotomic variables which involve random choices of A's and B's observables from a set of up to four elements.

In addition, some new results for lower bounds on higher numbers of outputs are presented. Finally, it is demonstrated that the 1-norm (used here as the distance) between the point defined by the observed probabilities

and the relevant local polytope represents a convenient way of quantifying violations of locality in the presence of experimental uncertainties. This distance can be determined in a straightforward way by linear programming.

This paper is organized as follows: Sec. II summarizes relevant and already known results on classical correlations, classical transfer functions, and their relation to probability polytopes. The local polytopes and nonlocal no-signaling polytopes are introduced. These describe classical local correlations, and classical nonlocal correlations which fulfill the no-signaling condition, respectively. It is shown how these polytopes can be represented in terms of their vertexes or equivalently in terms of inequalities for their facets.

In Sec. III these two types of polytope are used to determine lower bounds on minimum detection efficiencies which still allow for a violation of locality by quantum systems. Sec. IV finally demonstrates how the 1-norm defining the distance of a given probability distribution from the relevant local polytope can be determined with the help of linear programming.

II. CLASSICAL CORRELATIONS AND PROBABILITY POLYTOPES

In this section basic concepts involved in the description of classical bipartite correlations are summarized. For this purpose transfer functions and probability polytopes are introduced [8, 9, 10]. In particular, the local Bell polytope $\mathcal{L}$ and the nonlocal no-signaling polytope $\mathcal{P}$ are discussed in detail.

A. Transfer functions and transition probabilities

Given a classical deterministic system with discrete inputs x and outputs a, the output is a definite function F of the input: $a = F(x)$. Thereby the transfer function of the system, F, specifies a single transition from x to a for every input x. If the system may be stochastic, then the behavior of the system has to be described in terms of the transition probabilities $P(a|x)$, which define a point in a transition probability space whose coordinates are these probabilities. Since, for a given input, the total probability of an output must be unity, the probabilities satisfy the normalization condition $\sum_a P(a|x) = 1$. For a deterministic system with transfer function F the probabilities are $P(a|x) = \delta(a, F(x))$, with possible values 0 or 1. In terms of these particular probabilities an arbitrary transition probability of a stochastic system can be represented by [9]

$$P(a|x) \;=\; \sum_F P(F)\delta(a, F(x)) \tag{1}$$

with $P(F)$ denoting the probability with which the deterministic transfer function F governs the correlations under consideration.

If there are $N(x)$ possible values for the input x and $N(a)$ possible values for the output a, there are $N(a)^{N(x)}$ possible transfer functions, but only $N(x) \times N(a)$ transition probabilities, so usually there are many more transfer functions than there are transition probabilities and the expansion in terms of transfer functions is not generally unique. The sum of Eq.(1) is over all transfer functions, but if there are constraints on them, it can be over a subset of F.

A typical bipartite Bell experiment testing locality, such as the one described by Eberhard [6], involves a two-photon source distributing one photon to Alice (A) and the other photon to Bob (B). The subsequent experiment performed by A and B may be considered as an input-output system, in which A's input x is a choice of angle for the measurement of a photon polarization and her output a is the result of the measurement, $+$ or $-$, depending on whether the polarization is found to be parallel or perpendicular to the chosen angle. Similarly for B with input y and output b. In the simplest case A and B each have a choice of two angles only, a different pair for A and for B. So each of them has 2 inputs and 2 outputs, resulting in 4 inputs and 4 outputs for the whole system. In generalizations of bipartite Bell experiments the number of inputs as well as the number of possible outputs of A and B may also be larger. Notice that the outputs are classical events which result from quantum measurements. Since the transition probabilities are all probabilities of these classical events, the analysis of a Bell experiment does not depend in any way on quantum theory, although the design of such an experiment clearly does.

Assuming locality means that for deterministic systems A's output can only depend on her input, and the same for B. So a transfer function F for the whole system is made up of one transfer function for A and one for B: $F = (F^A, F^B)$, where $a = F^A(x)$ and $b = F^B(y)$. This is the locality constraint on transfer functions, which in general reduces their possible number significantly.

Thus if the numbers of possible inputs and outputs of A are denoted by $N(x)$ and $N(a)$ and of B by $N(y)$ and $N(b)$ respectively, the total number of local transfer functions is given by $N(a)^{N(x)} \times N(b)^{N(y)}$ and the number of corresponding local transition probabilities is given by $N(x) \times N(a) \times N(y) \times N(b)$, although the latter are not independent.

So for any classical theory of a Bell experiment, the transition probabilities must be obtainable from some local transfer function probabilities using a basic equation of the form

$$P(ab|xy) \;=\; \sum_{F^A, F^B} P(F^A, F^B)\delta(a, F^A(x))\delta(b, F^B(y)). \tag{2}$$

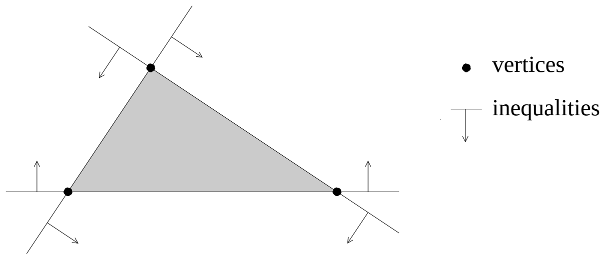

FIG. 1: The two possible representations of a polytope by its vertexes ($\mathcal{V}$-representation) and by inequalities characterizing half-spaces ($\mathcal{H}$-representation).

B. Probability polytopes and their representations

A convex polytope in a space of dimension D is a generalization of a convex polygon in 2-space or of a convex polyhedron in 3-space. It can be defined as all those points whose coordinates are a weighted sum of the coordinates of its vertexes, with non-negative weights that sum to unity. So by the fundamental equation (1), if we treat $P(F)$ as weights, the point given by the transition probabilities of a system with transfer functions F lies in the probability polytope whose vertexes are defined by these transfer functions. This defines the so called vertex- or $\mathcal{V}$-representation of the probability polytope.

There is an alternative representation of this probability polytope, the so called half-space- or $\mathcal{H}$-representation, in terms of a set of inequalities, each of which defines a half space. A central theorem of polytope theory [10] states that any polytope can always be described either by its $\mathcal{V}$-representation or by its $\mathcal{H}$-representation. Figure 1 illustrates this equivalence of the two possible representations schematically in the simple case of a triangle.

1. *Local probability (Bell) polytopes*

Let us consider the special case of local classical correlations in more detail. These correlations are of central interest for the analysis of Bell experiments that test locality. Provided there are $N(x)$ and $N(y)$ possible values of inputs x and y and $N(a)$ and $N(b)$ possible values for the outputs a and b of A and B, there are $N(x)\times N(y)\times N(a)\times N(b)$ possible local transition probabilities and $N(a)^{N(x)} \times N(b)^{N(y)}$ possible local transfer functions. Each of these transfer functions defines a vertex of the corresponding local probability or Bell polytope $\mathcal{L}$, thus yielding its $\mathcal{V}$-representation. Any physical system whose transition probabilities are located outside this Bell polytope is nonlocal.

Whereas the $\mathcal{V}$-representation of a Bell polytope can be obtained in a straightforward way from the transfer functions, the determination of its corresponding $\mathcal{H}$-representation is a considerably more difficult numerical problem for polytopes with large numbers of vertexes [11]. The half-spaces of the $\mathcal{H}$-representation of a Bell polytope can be divided into several classes. The least interesting class of inequalities expresses the non-negativity conditions of all probabilities involved, i.e.

$$P(ab|xy) \geq 0 \quad \forall\,(abxy), \tag{3}$$

and the corresponding normalization conditions, i.e.

$$\sum_{ab} P(ab|xy) = 1 \quad \forall\,(xy)\,. \tag{4}$$

A more interesting class of constraints are the *no-signaling equalities*,

$$\begin{aligned}\sum_a P(ab|x_1y) &= \sum_a P(ab|x_2y) \quad \forall\,(x_1x_2yb),\\ \sum_b P(ab|xy_1) &= \sum_b P(ab|xy_2) \quad \forall\,(xy_1y_2a),\end{aligned} \tag{5}$$

which follow because for a local system no signals may be sent from A to B or from B to A.

The third and most interesting inequalities of the $\mathcal{H}$-representation are the Bell inequalities themselves. Because the transformation from $\mathcal{V}$- to $\mathcal{H}$-representation is difficult, they are known only in special cases, so the search for new families of Bell inequalities is still an active research area [11, 12, 13, 14].

Our subsequent discussion will focus on the $\mathcal{V}$-representation of Bell polytopes, as they can be determined from the relevant transfer functions in a straightforward way. Furthermore, by taking into account constraints on the transition probabilities arising from conservation of probability and from locality, the $\mathcal{V}$ -representation of Bell polytopes can be obtained efficiently in a reduced basis. This fact was realized earlier already by Pitovski [15, 16, 17]. Let us start from the observation that the full space of local transition probabilities is $N(x)\times N(a)\times N(y)\times N(b)$-dimensional. Due to conservation of probability these transition probabilities fulfill the $N(x)\times N(y)$ relations

$$\sum_{a,b} P(ab|xy) = 1. \tag{6}$$

Thus, for each choice of input (x, y) by A and B one output, say $(a_{N(a)}, b_{N(b)})$, can be eliminated by this linear dependence. Furthermore, if A's output is equal to $a_{N(a)}$ all joint transition probabilities involving this output can be expressed as

$$P(a_{N(a)}b|xy) = P_B(b|xy) - \sum_{a\neq a_{N(a)}} P(ab|xy), \tag{7}$$

where $P_B(b|xy) = \sum_a P(ab|xy)$ is B's marginal transition probability. Because of the no-signaling constraints (5) this marginal transition probability cannot depend on A's choice of input x, i.e. $P_B(b|xy) \equiv P_B(b|y)$. An analogous argument applies to B, i.e.

$$P(ab_{N(b)}|xy) = P_A(a|xy) - \sum_{b\neq b_{N(b)}} P(ab|xy) \tag{8}$$

with $P_A(a|xy) \equiv P_A(a|x)$. Thus, the marginal and joint transition probabilities which do not contain the outputs $a_{N(a)}$ or $b_{N(b)}$ are linearly independent and span the full space of local transition probabilities of a Bell polytope, so they form a basis of dimension

$$\begin{aligned} D &= N(x) \times (N(a) - 1) + N(y) \times (N(b) - 1) + \\ &\quad N(x) \times N(y) \times (N(a) - 1) \times (N(b) - 1) \end{aligned} \tag{9}$$

for the Bell polytope. As a result the Bell polytope is given by all these linearly independent marginal and joint transition probabilities fulfilling the conditions

$$\begin{aligned} \sum_{a \neq a_{N(a)}} P_A(a|x) &\leq 1, \quad \sum_{b \neq b_{N(b)}} P_B(b|y) \leq 1, \\ P(ab|xy) &= P_A(a|x) P_B(b|y) \end{aligned} \tag{10}$$

for $a \neq a_{N(a)}, b \neq b_{N(b)}$ and all possible $N(x) \times N(y)$ inputs. Furthermore, the vertexes of the Bell polytope are given by all those points in this probability space whose coordinates assume the values 0 and 1 only and which are consistent with relations (10).

2. Nonlocal no-signaling probability polytopes

In an experiment with detection efficiency close to the ideal, it would be possible to demonstrate the correlations of weak nonlocality, but so far there has always been at least one loophole [18]. Further, Bell's theorem itself is incomplete, since it is based on the unproved assumption that such detection is possible [19, 20].

Thus, it is of interest to determine minimum detection efficiencies which still allow for a violation of locality by quantum theory provided the local measurements of A and B are separated by a spacelike interval. For the determination of these minimum detection efficiencies a detailed knowledge of the set of correlations produced by entangled quantum systems is required. Unfortunately, a complete characterization of all possible correlations of bipartite local measurements of quantum systems does not yet exist [21].

However, for any given number of inputs and outputs nonlocal no-signaling polytopes $\mathcal{P}$ can be constructed which include all possible bipartite quantum correlations of quantum systems of arbitrary dimensions. Therefore, the boundary of the region representing all possible bipartite quantum correlations is sandwiched between the boundaries of a nonlocal no-signaling polytope $\mathcal{P}$ and its corresponding Bell polytope. For any given set of inputs and outputs, this enables one to obtain lower bounds on minimum detection inefficiencies which still allow the observation of nonlocal features of quantum systems and which are independent of the dimension of the quantum system and the associated choice of quantum observables generating these correlations.

In the case of $N(x) \times N(y)$ inputs and $N(a) \times N(b)$ outputs, the associated nonlocal no-signaling polytope is defined by all joint transition probabilities $P(ab|xy)$ which are constrained only by the no-signaling conditions (5). It should be stressed that these no-signaling conditions are weaker than locality because they do not necessarily imply that the underlying transfer functions are local.

Thus, in general the no-signaling conditions are also compatible with transfer functions of the form $F \neq (F^A, F^B)$. As with the local Bell polytopes of Sec. II B 1, the nonlocal no-signaling polytopes can be described conveniently in the reduced basis formed by all marginal and joint transition probabilities which do not contain outputs $a_{N(a)}$ or $b_{N(b)}$ and whose dimension is given by relation (9). In this reduced basis the no-signaling constraints (5) are already taken into account provided the marginal and joint transition probabilities fulfill the consistency constraints

$$\begin{aligned} \sum_{a \neq a_{N(a)}} P(ab|xy) &\leq P_B(b|y), \\ \sum_{b \neq b_{N(b)}} P(ab|xy) &\leq P_A(a|x) \end{aligned} \tag{11}$$

for all inputs (x, y). These inequalities follow from Eqs. (7) and (8) and the no-signaling constraints (5). So the requirement of no-signaling is weaker than locality. Furthermore, these inequalities indicate that the nonlocal no-signaling polytopes are defined in a natural way in the $\mathcal{H}$-representation. Thus, for large dimensions of the reduced basis obtaining the corresponding $\mathcal{V}$-representation from this $\mathcal{H}$-representation is a difficult numerical problem that limits the number of inputs and outputs considerably for which this conversion can be achieved.

III. DETECTION INEFFICIENCIES

Based on the previously discussed local and no-signaling polytopes $\mathcal{L}$ and $\mathcal{P}$, in this section lower bounds on minimum detection efficiencies are obtained below which violations of locality cannot be observed in bipartite Bell experiments. The dependence of these lower bounds on the numbers of inputs and outputs and on symmetry is explored. It should be emphasized that for a given number of inputs and outputs these lower bounds on minimum detection efficiencies apply to correlations originating from arbitrary bipartite quantum states and observables of arbitrary dimensional Hilbert spaces. A related problem, namely the determination of maximum possible values of detection efficiencies which still guarantee locality, has recently been investigated by Bigelow [22] with the help of linear programming techniques for some special cases of correlations originating from two- and three-qubit systems. Contrary to our approach this investigation does not involve the nonlocal no-signaling polytope so that its resulting conclusions apply only to correlations which originate from two- and three-qubit quantum systems and from particular choices of quantum observables.

One of the simplest ways to describe detection inefficiencies of A and B is by a parameter $\eta \in [0,1]$ describing the total efficiency of the detection systems involved. Thus, η is the probability that a detector fires if it actually should. In practice, detection inefficiencies can have different physical origins. They can originate from an imperfect photodetector, for example, which does not respond to each photon hitting its detection surface. Alternatively, they may also arise from the fact that due to the particular geometry of an experimental setup only a fraction of photons propagating within a small solid angle is capable of hitting a photodetector at all. In the following we assume that a combination of these effects gives rise to the detection efficiencies η_1 and η_2 of A and B in a bipartite Bell experiment. Furthermore, these detection efficiencies are assumed to be independent of the polarization of the photons hitting the photodetectors.

For ideal detection, in a dichotomic Bell experiment A always receives a photon, so she needs only one detector to distinguish the polarizations. But for real detectors, a single polarization-sensitive detector makes no distinction between the absence of a photon and a photon with the wrong polarization, whereas two polarization-sensitive detectors can distinguish between these two cases. Similarly for B. Thus, in imperfect situations two detectors give output events that are not possible with only one, increasing the dimension of the relevant polytopes. Indeed, we will demonstrate that the two cases can give rise to different lower bounds on detection efficiencies.

First of all let us describe detection inefficiencies where A's (B's) detector cannot distinguish between the no-detection event and the event $a_{N(a)}$ ($b_{N(b)}$)). Thus, the ideal joint transition probabilities, $P_{1,1}(ab|xy)$, are related to the corresponding imperfect joint transition probabilities, $P_{\eta_1,\eta_2}(ab|xy)$, by

$$\begin{aligned}
P_{\eta_1,\eta_2}(ab|xy) &= \eta_1\eta_2 P_{1,1}(ab|xy), \\
P_{\eta_1,\eta_2}(ab_{N(b)}|xy) &= \eta_1 P_{1,1}(ab_{N(b)}|xy) + \\
&\quad \eta_1(1-\eta_2)\sum_{b\neq b_{N(b)}} P_{1,1}(ab|xy), \\
P_{\eta_1,\eta_2}(a_{N(a)}b|xy) &= \eta_2 P_{1,1}(a_{N(a)}b|xy) + \\
&\quad \eta_2(1-\eta_1)\sum_{a\neq a_{N(a)}} P_{1,1}(ab|xy), \\
P_{\eta_1,\eta_2}(a_{N(a)}b_{N(b)}|xy) &= \eta_1\eta_2 P_{1,1}(a_{N(a)}b_{N(b)}|xy) + \\
&\quad \eta_1(1-\eta_2)\sum_{b} P_{1,1}(a_{N(a)}b|xy) + \\
&\quad \eta_2(1-\eta_1)\sum_{a} P_{1,1}(ab_{N(b)}|xy) + \\
&\quad (1-\eta_1)(1-\eta_2) \qquad (12)
\end{aligned}$$

for all outputs $(a \neq a_{N(a)}, b \neq b_{N(b)})$ and inputs (x,y) of A and B. For dichotomic Bell experiments with photons this describes situations in which A and B each use one polarization-sensitive photodetector only which cannot distinguish between a photon with the wrong polarization and a no-detection event. In the reduced basis of marginal and joint transitions probabilities discussed in Secs. II B 1 and II B 2, in which the outputs $a_{N(a)}$ and $b_{N(b)}$ are eliminated, these relations reduce to the simple form

$$\begin{aligned}
P_{A\eta_1\eta_2}(a|x) &= \eta_1 P_{A1}(a|x),\ P_{B\eta_1\eta_2}(b|y) = \eta_2 P_{B1}(b|y), \\
P_{\eta_1\eta_2}(ab|xy) &= \eta_1\eta_2 P_{1,1}(ab|xy) \qquad (13)
\end{aligned}$$

for all outputs $(a \neq a_{N(a)}, b \neq b_{N(b)})$ and inputs (x,y) of A and B.

If, in contrast, the no-detection event $\emptyset$ is treated as an additional output, the dimension of the relevant transition probability polytope is increased. In this case the ideal and imperfect transition probabilities $P_{1,1}(ab|xy)$ and $P_{\eta_1\eta_2}(ab|xy)$ are related by

$$\begin{aligned}
P_{\eta_1,\eta_2}(ab|xy) &= \eta_1\eta_2 P_{1,1}(ab|xy), \\
P_{\eta_1,\eta_2}(a\emptyset|xy) &= \eta_1(1-\eta_2)\sum_{b} P_{1,1}(ab|xy), \\
P_{\eta_1,\eta_2}(\emptyset b|xy) &= (1-\eta_1)\eta_2\sum_{a} P_{1,1}(ab|xy), \\
P_{\eta_1,\eta_2}(\emptyset\emptyset|xy) &= (1-\eta_2)(1-\eta_1) \qquad (14)
\end{aligned}$$

for all outputs (a,b) and inputs (x,y) of A and B. For dichotomic Bell experiments with photons this describes situations in which A and B each use two photodetectors which are sensitive to two orthogonal polarizations. By eliminating from Eqs. (14) all joint transition probabilities involving the outputs $a_{N(a)}$ or $b_{N(b)}$ with the help of the marginal transition probabilities one obtains the corresponding relations between the ideal and imperfect transition probabilities of the reduced basis.

For a given number of inputs and outputs, lower bounds on detection efficiencies below which a violation of locality is no longer possible can be obtained from Eqs. (12), (13), and (14) by identifying the ideal transition probabilities $P_{1,1}(ab|xy)$ with the possible correlations of the nonlocal no-signaling polytope $\mathcal{P}$ and by determining the critical detection efficiencies (η_1, η_2) at which the corresponding imperfect transition probabilities $P_{\eta_1,\eta_2}(ab|xy)$ merge into the Bell polytope $\mathcal{L}$.

These critical detection efficiencies η_1 and η_2 determine lower bounds on the detection efficiencies below which a violation of locality is no longer possible by the corresponding correlations produced by any quantum system. In general, it is unclear whether the lower bounds obtained on the basis of the no-signaling polytope $\mathcal{P}$ can be reached by any quantum system with appropriate choices of the dimension of the Hilbert space and of the quantum observables. But it is shown later that in the special case of two inputs and two outputs of both A and B these lower bounds actually can be reached.

Table I summarizes numerically-determined lower bounds on detection efficiencies (η_1 and η_2) of A and B which characterize the merging of the imperfect transition probabilities P_{η_1,η_2} into the relevant local polytope $\mathcal{L}$. If A and B randomly choose one of two possible physical variables in their respective laboratories,

(A,B) # inputs	$\eta_1=\eta_2$ add. outcome	$\eta_1=\eta_2$ no add. outcome	$\eta_1=1$ add. outcome	$\eta_1=1$ no add. outcome
$(2,2)$	0.6667	0.6667	0.5000	0.5000
$(3,2)$	0.6667	0.6667	0.5000	0.5000
$(3,3)$	0.5714	0.6000	0.3333	0.3333
$(4,3)$	0.5000	0.5714	0.2500	0.2500

TABLE I: Critical detection efficiencies, η_1 and η_2, of Alice (A) and Bob (B) for dichotomic bipartite symmetric ($\eta_1 = \eta_2$) and extreme asymmetric ($\eta_1 = 1$) Bell experiments with various numbers of inputs and for cases in which the no-detection event is treated separately (add. outcome) and in which it is combined with the event (a_2, b_2) (no add. outcome).

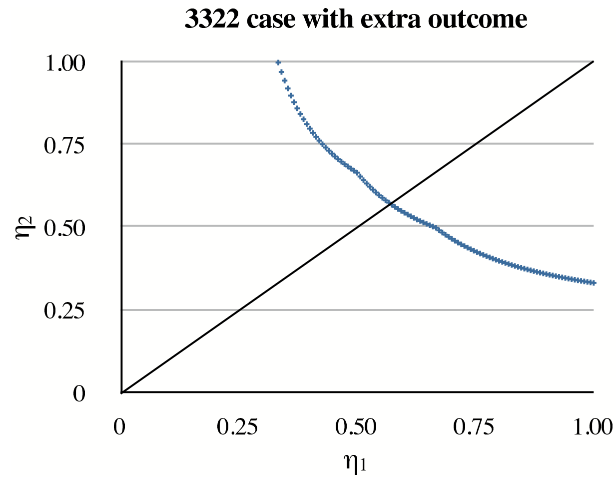

FIG. 2: Lower bounds on detection efficiencies (η_1, η_2) for three inputs on Alice's and Bob's sides with the no-signal event treated as an extra output.

i.e. case (2,2), and if they have identical detectors, i.e. $\eta \equiv \eta_1 = \eta_2$ (symmetric case), the resulting critical value of η turns out to be independent of whether or not the no-detection event is treated as an additional output. This optimal lower bound obtained on the basis of the no-signaling polytope $\mathcal{P}$ turns out to be identical with the minimal detection efficiency obtained previously by Eberhard [6]. Eberhard's result demonstrated that pure two-qubit quantum states exist which are able to violate locality down to minimum detection efficiencies of magnitude $\eta = 0.6667$ in symmetric cases. Surprisingly these most robust two-qubit quantum states are almost separable. Also the optimal lower bounds for the corresponding extreme asymmetric cases of Table I, i.e. $\eta_1 = 1 \neq \eta_2$ of magnitude $\eta_2 = 0.5000$, in which Alice's detection efficiency is assumed to be perfect, are independent of whether or not the no-detection event is treated as an additional output. These lower bounds also agree with the minimum possible detection efficiencies of Eberhard [6] which just allow for a violation of locality by two-qubit quantum systems.

It should be mentioned that apart from reproducing Eberhard's previous minimum detection efficiencies our results of Table I for the $(2,2)$-case also demonstrate that there is no way to violate locality with detection efficiencies below $\eta = 0.667$ in the symmetric case and below $\eta_2 = 0.500$ in the extreme asymmetric case. This conclusion holds for arbitrary choices of two two-valued quantum observables of A and B and for arbitrary bipartite quantum states in arbitrary dimensional Hilbert spaces, which produce the statistical correlations.

Table I also includes results on optimal lower bounds for cases in which more than two physical variables are selected on Alice's or Bob's sides. It is apparent that in general these lower bounds depend on whether or not the no-detection event is treated as an additional output. Furthermore, the lower bounds of cases in which the no-detection event is treated as an additional output are always lower than or equal to cases in which the no-detection event is combined with an already existing output. However, for cases with more than two inputs of Alice or Bob it is not known yet whether quantum systems exist which are capable of violating locality all the way down to these lower bounds. However, the number of outputs on Alice's and Bob's sides, $N(a)$ and $N(b)$, puts a lower bound on the dimension D of the Hilbert space of these quantum systems, i.e. $D \geq N(a) \times N(b)$.

In Figs. 2 and 3 lower bounds on minimum detection efficiencies (η_1, η_2) are depicted for arbitrary cases between the symmetric ($\eta_1 = \eta_2$) and the extreme asymmetric ($1 = \eta_1 \neq \eta_2$) situation for the special case of three inputs and two outputs of both A and B. Irrespective whether or not the no-signal event is treated as a separate outcome one observes a cusp-like dependence in these figures. This non-smooth dependence corresponds to a case in which, at a particular value of (η_1, η_2), a vertex of the properly transformed nonlocal no-signaling polytope (compare with Eqs.(12)) just coincides with a vertex of the local (Bell) polytope $\mathcal{L}$.

We have also explored lower bounds on detection efficiencies for two inputs and three outputs on both Alice and Bob's sides. In the symmetrical case ($\eta_1 = \eta_2 \equiv \eta$) the lower bound was given by $\eta = 0.6667$, whether or not the no-signal event was combined with an output. Similarly for the extreme asymmetric case ($\eta_1 = 1$) we obtained the lower bound $\eta_2 = 0.5000$.

It is difficult to determine lower bounds on detection efficiencies numerically with the help of the no-signaling polytope $\mathcal{P}$ for larger numbers of inputs or outputs. This is due to the fact that no-signaling polytopes are defined in a natural way in the $\mathcal{H}$-representation (compare with the discussion of Sec. II B 2). Thus in order to determine lower bounds on detection efficiencies one has to convert the nonlocal no-signaling polytope from its $\mathcal{H}$-representation into its $\mathcal{V}$-representation, which becomes very difficult numerically for such cases.

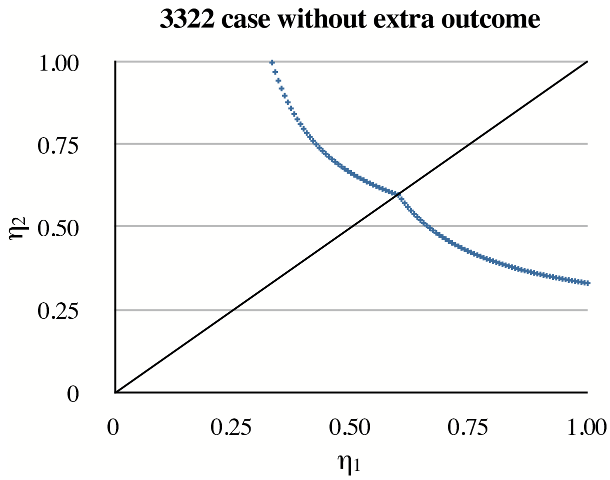

FIG. 3: Lower bounds on detection efficiencies (η_1, η_2) for three inputs on Alice's and Bob's sides with the no-signal event combined with the output (a_2, b_2).

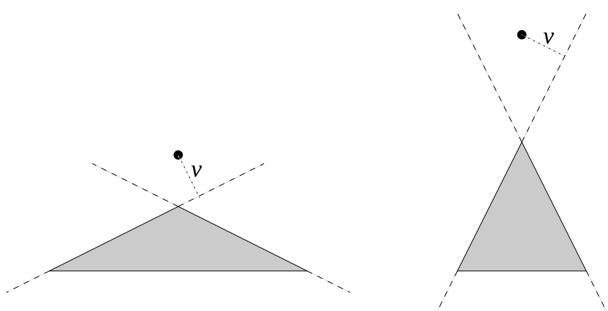

FIG. 4: Schematic representations of identical violations v of relevant Bell inequalities. Note the different distances from the polytopes.

IV. DISTANCE MEASURES QUANTIFYING THE VIOLATION OF LOCALITY

The simplest way of testing correlations for locality is to determine whether or not the relevant point $\mathbf{X}$ of transition-probability space is inside the local polytope $\mathcal{L}$. However, in general probabilities can be estimated experimentally only approximately with an uncertainty depending on the size of the statistical sample involved. Thus, a practically useful method for determining violations of locality should also quantify how far outside $\mathcal{L}$ the given point $\mathbf{X}$ is located. Therefore, it is desirable to develop methods which permit one to find the distance of a given point $\mathbf{X}$ in transition-probability space from a local polytope $\mathcal{L}$, so that one can decide whether observed transition probabilities with their experimental uncertainties still violate locality.

A natural choice for such a distance measure is the Euclidean 2-norm distance from the nearest facet of the polytope $\mathcal{L}$ which corresponds to a Bell inequality. But also the 1- or the ∞-norms are possible choices. However, as illustrated in Fig. 4, it definitely makes more sense to consider the "distance of $\mathbf{X}$ from the polytope $\mathcal{L}$ as a whole". The only reasonable definition for the "distance from a polytope" is the minimum distance of any point of $\mathcal{L}$ to $\mathbf{X}$, which means we have to deal with an optimization problem.

Let us assume $\mathcal{L}$ is given in D-dimensional space and that it has r vertexes, which we denote by $\mathbf{v}_i \in \mathbb{R}^D$. Thus, any point $\mathbf{Y} \in \mathcal{L}$ can be written as a convex combination of these vertexes, i.e. $\mathbf{Y} = \sum_{i=1}^r w_i \mathbf{v}_i$ with weights $w_i \geq 0$ and with $\sum_{i=1}^r w_i = 1$. So, it is natural to use the w_i (or the vector $\mathbf{w}^T = (w_1; w_2; \ldots; w_r)$) as the coordinates for the optimization problem rather than the coordinates of $\mathbf{Y}$ in the actual space $\mathbb{R}^D$ in which the polytope lives. This is motivated by the fact that the constraints of the polytope are given in terms of the weights w_i rather than in terms of the coordinates of the actual space. However, we still want to optimize the distance in the actual space. In order to achieve this for the 1-norm, it is convenient to introduce the matrix $\mathbf{C} = (\mathbf{v}_1; \mathbf{v}_2; \ldots; \mathbf{v}_r) \in \mathbb{R}^{D\times r}$ with $\mathbf{Y} = \mathbf{C}\cdot\mathbf{w}$. Let us also use the abbreviations $\mathbf{0}_D$, $\mathbf{1}_D$ and $\mathbf{0}_r$, $\mathbf{1}_r$ for column vectors of all zeros or ones in $\mathbb{R}^D$ and $\mathbb{R}^r$, respectively. Analogously we use the notation $\mathbf{1}_{D\times D}$ for a diagonal $D\times D$ unit matrix, and similarly $\mathbf{1}_{r\times r}$. The problem of finding the minimum distance between a point $\mathbf{X} \in \mathbb{R}^D$ and the local polytope $\mathcal{L}$ can now be formulated as the following linear programming problem:

$$\begin{aligned}&\text{Maximize } -(\mathbf{1}_D^T; \mathbf{0}_r^T)\cdot\mathbf{Z}\\ &\text{Subject to } \mathbf{A}\cdot\mathbf{Z} \leq \mathbf{b}\end{aligned}$$

with the $(2D+3)\times(D+r)$ matrix

$$\mathbf{A} = \begin{pmatrix} -\mathbf{1}_{D\times D} & , & \mathbf{C} \\ -\mathbf{1}_{D\times D} & , & \mathbf{C} \\ \mathbf{0}_D^T & , & -\mathbf{1}_{r\times r} \\ \mathbf{0}_D^T & , & \mathbf{1}_r^T \\ \mathbf{0}_D^T & , & -\mathbf{1}_r^T \end{pmatrix}, \tag{15}$$

the $(D+r)$-dimensional vector $\mathbf{Z}^T = (\bar{\mathbf{Z}}^T; \mathbf{w}^T)$ and the $(2D+r+2)$-dimensional vector $\mathbf{b}^T = (\mathbf{X}^T; -\mathbf{X}^T; 0; 1; ; 1)$. As a result the 1-norm is given by $(\mathbf{1}_D^T; \mathbf{0}_r^T)\cdot\mathbf{Z} \equiv \mathbf{1}_D^T\cdot\bar{\mathbf{Z}}$.

A similar linear programming problem can be formulated in order to find the ∞-norm. Although an analogous quadratic programming problem can be formulated for the ordinary 2-norm distance, it is worth mentioning that the numerical solution of this quadratic problem is much more difficult and time-consuming than the corresponding linear programming problem. The 2-norm can however be bounded from above and below by the 1-norm and ∞-norm, respectively. As an example, consider Fig. 5 which shows how the distance between a properly transformed vertex of the nonlocal no-signaling polytope $\mathcal{P}$ (compare with Eqs. (12)) and the local polytope $\mathcal{L}$ varies smoothly when we vary the detection efficiency. Of course, at $\eta = 0.6667$ all of the distance measures vanish.

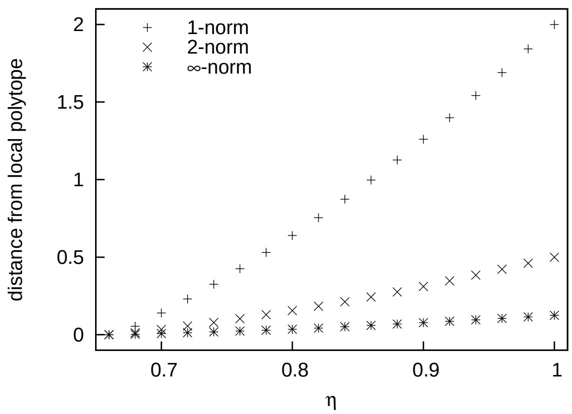

FIG. 5: Distance of a vertex of $\mathcal{P}$ from $\mathcal{L}$ as a function of η for two inputs and outputs of both A and B.

V. SUMMARY AND CONCLUSIONS

For given numbers of inputs and outputs we have investigated minimum detection efficiencies below which locality cannot be violated by correlations produced by any quantum system in bipartite Bell experiments. For this purpose lower bounds on these minimum detection efficiencies have been obtained numerically with the help of linear programming techniques. Our determination of these lower bounds is based on the observation that for any given number of inputs and outputs any possible bipartite correlation produced by a quantum system in an arbitrary dimensional Hilbert space is sandwiched between the boundaries of the nonlocal no-signaling polytope and the Bell polytope. Thus, for imperfect detection the detection efficiencies at which statistical correlations of the properly transformed nonlocal no-signaling polytope merge into the Bell polytope yield lower bounds on these minimum detection efficiencies.

Both the local (Bell) and the nonlocal no-signaling polytope can can be dealt with conveniently by linear programming. In particular, the vertex representation of any Bell polytope can be determined in a straightforward way. The construction of the nonlocal no-signaling polytope is more complicated as it is naturally defined in the $\mathcal{H}$-representation.

Our numerically calculated lower bounds on detection efficiencies demonstrate that in general, with the exception of two inputs and outputs of A and B, these bounds are not identical for Bell experiments with symmetric and asymmetric detection efficiencies. Furthermore, in the case of two inputs and outputs our lower bounds agree with the minimum detection efficiencies obtained previously by Eberhard [6] for two-qubit quantum correlations. Thus, in this case our results demonstrate that these minimum detection efficiencies cannot be lowered even if one considered quantum correlations originating from quantum systems of arbitrary dimensions.

Our investigation constitutes a first step towards a systematic study of bipartite correlations produced by quantum systems. In general, it is still unclear to what extent our numerically determined lower bounds can be reached by correlations of appropriately chosen quantum systems. Further research is required to clarify this point. In addition, for numerical purposes it would be desirable to find an effective way for determining directly the $\mathcal{V}$-representations of nonlocal no-signaling polytopes without involvement of their $\mathcal{H}$-representations. This would also allow the efficient treatment of cases involving many inputs and outputs. Furthermore, our approach can also be adapted to the investigation of multipartite correlations.

Acknowledgments

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[1] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, A. Zeilinger, Phys. Rev. Lett. **81**, 5039 (1998).
[2] N. Gisin, arXiv preprint quant-ph/0702021.
[3] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge UP, Cambridge, 1987).
[4] A. Cabello, J.-A. Larsson, Phys. Rev. Lett. **98**, 220402 (2007).
[5] N. Brunner, N. Gisin, V. Scarani, C. Simon, Phys. Rev. Lett. **98**, 220403 (2007).
[6] P. H. Eberhard, Phys. Rev. A **47**, R747 (1993).
[7] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, D. Roberts, Phys. Rev. A **71**, 022101 (2005).
[8] I. C. Percival, Phys. Lett. A **244**, 495 (1998).
[9] I. C. Percival, arXiv preprint quant-ph/9906005.
[10] G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics **152** (Springer, Berlin, 1995).
[11] R. E. Werner, M. M. Wolf, Phys. Rev. A **64**, 32112 (2001).
[12] S. Massar, S. Pironio, J. Roland, B. Gisin, Phys. Rev. A **66**, 52112 (2002).
[13] D. Collins, N. Gisin, J. Phys. A: Math. Gen. **37**, 1775 (2004).
[14] D. Avis, H. Imai, T. Ito, Y. Sasaki, J. Phys. A: Math. Gen. **38**, 10971 (2005).
[15] I. Pitovski, *Quantum Probability – Quantum Logic*, Lecture Notes in Physics **321** (Springer, Berlin, 1989).
[16] I. Pitovski, Math. Prog. **50**, 392 (1991).
[17] I. Pitovski, K. Svozil, Phys. Rev. A **64**, 014102 (2001).
[18] M. Aspelmeyer, A. Zeilinger, Physics World **21** No **7**, 22 (2008).
[19] I. C. Percival, B. Garraway, Phys. Lett. A **370**, 213 (2007).
[20] P. A. M. Dirac, *The Principles of Quantum Mechanics*, 3rd edn, p. 34–37 (Oxford UP, Clarendon, 1947).

[21] M. Navascues, S. Pironio, A. Acin, Phys. Rev. Lett. **98**, 010401 (2007) (see also arXiv peprint 0803.4290).

[22] J. A. Bigelow, arXiv peprint 0805.0387.